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Fractional part sums and divisor functions II

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§ 1. Introduction

This is a continuation of Part I [1] and deal with the negative power case. That is , we consider $(t>0)$

$$\sum_{n\leq x} n^{-t}\sigma_a(n) \quad , \quad \sum_{n\leq x} E^a_{-t}(n) \quad , \quad \int_1^x E^a_{-t}(u)du \ ,$$

where

$$\sigma_a(n)=\sum_{d\mid n} d^a \quad ,$$

$E^a_{-t}(x)$: the error term associated with $\sum_{n\leq x} n^{-t}\sigma_a(n)$.

Our theorem generalize and refine and in some cases correct MacLeod's theorem [3].

Let $B_k(x)$ denote the k-th Bernoulli polynomial, $[x]$ the integral part of x , $\overline{B}_k(x) = B_k(x-[x])$ the k-th periodic Bernoulli polynomial, and define the basic function $G_{a,k}(x)$ by

$$G_{a,k}(x)=\sum_{n\leq\sqrt{x}} n^a\,\overline{B}_k\left(\frac{x}{n}\right) \quad ,$$

for $a\in\mathbb{R}$, $k\in\mathbb{N}$.

Our aim amounts to writing the error terms associated with above three sums explicitly in terms of the function $G_{a,k}(x)$, and get informations on the behaviour of the error terms and infer new results on $G_{a,k}(x)$.

§ 2. Statement of results

Theorem 1 If $a>0$ and $t \leqq a$, $a-t \in \mathbb{Z}$, then we have

$$\sum_{n\leqq x} n^{-t}\sigma_a(n) = \frac{\zeta(a+1)}{a-t+1}x^{a-t+1} + \sum_{r=1}^{a-t+1}\frac{(-1)^r}{r}\binom{a-t}{r-1}x^{a-t+1-r}G_{r-a-1,r}(x)$$

$$+\begin{cases} \zeta(t-a)\zeta(t) - x^{-t}G_{a,1}(x), & t \geqq 2, \\ -\dfrac{x^{1-t}}{a(a-t+1)} + \begin{cases} \dfrac{\zeta(t-a)}{1-t}x^{1-t} + \zeta(t-a)\zeta(t), & t \neq 1, \\ \zeta(t-a)\log x + \gamma\zeta(t-a), & t=1, \end{cases} \\ + x^{1-t}\displaystyle\sum_{\substack{\max\{a-1,0\}<r \\ \leqq a-t+1}}\frac{(-1)^r}{r}\binom{a-t}{r-1}E_{r-a-1,r} - x^{-t}G_{a,1}(x), & 0<t<2, \end{cases}$$

$$+ O(x^{-t+a/2}),$$

where the constants $E_{r-a-1,r}$ are given in Lemma11[1], $\zeta(s)$ denotes the Riemann zeta-function, and γ Euler constant.

For the case $t>a$ we state it in Theorem 3 , since it needs a different argument.

For the second sum, under the same assumptions as Th.1, we distinguish 8 cases according to the values of a,t. Here we state the 1st and 5th cases.

Let $\sum_{n\leqq x} n^{-t}\sigma_a(n) = g^a_{-t}(x) + E^a_{-t}(x)$, where we define $E^a_{-t}(x)$ by

$$g^a_{-t}(x) = \begin{cases} \dfrac{\zeta(a+1)}{a}\, x^a + \zeta(t-a)\log x , & t=1,\ a \leqq 2 , \\[2ex] \dfrac{\zeta(a+1)}{a-t+1}\, x^{a-t+1} , & \text{otherwise.} \end{cases}$$

<u>Theorem 2</u>

$$\sum_{n\leqq x} E^a_{-a}(n) = \begin{cases} \dfrac{\zeta(1-a)}{(1-a)(2-a)}\, x^{2-a} + \dfrac{\zeta(a+1)-\zeta(a)}{2}\, x + (1/2 - \overline{B}_1(x))\dfrac{\zeta(1-a)}{1-a}\, x^{1-a} , & 0\leqq a=t<1, \\[2ex] \dfrac{\zeta(2)-\log 2\pi-\gamma}{2}\, x - \log x/4 - (1/2 - \overline{B}_1(x))G_{-1,1}(x) , & a=t=1 , \\[2ex] \dfrac{\zeta(1-a)}{(1-a)(2-a)}\, x^{2-a} + \dfrac{\zeta(a+1)-\zeta(a)}{2}\, x + \dfrac{\zeta(a-1)+\zeta(a+1)}{12} & \\[1ex] -1/2(1/2 - \overline{B}_1(x))\zeta(a) - (1/2 - \overline{B}_1(x))G_{-a,1}(x), & 1<a=t<2, \\[2ex] \dfrac{\zeta(3)-\zeta(2)}{2}\, x + \log x/12 + \gamma/12 + \zeta(3)/12 - \zeta'(-1) & \\[1ex] -1/2(1/2 - \overline{B}_1(x))\zeta(2) - (1/2 - \overline{B}_1(x))G_{-2,1}(x), & a=t=2 , \end{cases}$$

$$- G_{a+1,2}(x)/2x^a - G_{1-a,2}(x)/2 + O(x^{-a/2 + 1/2}) .$$

The 5th case is

$$\sum_{n\leqq x} E^1_{-1}(n) = \zeta(3)x^2/4 - (-1/6 + \gamma/12 - \zeta'(-1) + \zeta(3)\overline{B}_2(x)/2 + (1/2 - \overline{B}_1(x))G_{-2,1}(x))x$$

$$\frac{x}{2}\, G_{-1,2}(x) - \frac{1}{2x}\, G_{3,2}(x) + O(x^{1/2}) .$$

The corresponding MacLeods Th.8(vi) should be corrected as this .

Applying the best known result of Peng-Recknagel[4], we have

$$\sum_{n\leq x} E_{-1}^{1}(n) = \frac{\zeta(2)-\log 2\pi-\gamma}{2}\,x - \frac{1}{2}G_{0,2}(x) + O(x^{1/4})$$

$$= \frac{\zeta(2)-\log 2\pi-\gamma}{2}\,x + O(x^{2/7+\varepsilon}) \quad ,\forall\varepsilon>0 .$$

For the formula for the last integral, it is from Segal's formula[5], and Theorem 4-(c) of MacLeod[3].

In the end , we state

<u>Theorem 3</u> We restrict ourselves to the case $0 \leq a \leq 2$, $t=a+1$.

$$\sum_{n\leq x} n^{-a-1}\sigma_a(n) = \begin{cases} \frac{1}{2}\log^2 x + 2\gamma\log x + \gamma^2 + 2\gamma_1 , & a=0, \\ \zeta(a+1)\log x + \zeta(a+1) + \gamma\zeta\,(a+1) , & a>0, \end{cases}$$

$$- x^{-1}G_{-a,1}(x) - x^{-a-1}G_{a,1}(x)$$

$$+\begin{cases} 0 , & a=0, \\ -\frac{\zeta(1-a)}{a}\,x^{-a} , & 0<a<2, \\ 0 , & a=2, \end{cases}$$

$$+ O(x^{-1-a/2}) ,$$

where γ is Euler constant, $\gamma_1 = \lim_{x\to\infty}\left(\sum_{n\leq x} n^{-1}\log n - (\log^2 x)/2\right)$ the 1st generalized Euler constant.

In particular

$$\sum_{n\leq x}\frac{d(n)}{n} = \frac{1}{2}\log^2 x + 2\gamma\log x + \gamma^2 + 2\gamma_1 - \frac{2}{x}G_{0,1}(x) + O(x^{-1})$$

holds.

It is known that solving the Dilichlet's divisor problem is equivalent to verifying

$$G_{0,1}(x) = O(x^{1/4+\varepsilon}) \quad , \quad \forall\varepsilon > 0 .$$

Using the best known estimate due to Kolesnik[2], we have

$$\sum_{n\leq x}\frac{d(n)}{n} = \frac{1}{2}\log^2 x + 2\gamma\log x + \gamma^2 + 2\gamma_1 + O(x^{-73/108}\log^{\varepsilon} x)) \ , \ \forall\varepsilon > 0 .$$

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